The Forward Rate Cuve

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We will define the forward rate to be the time zero expection of the market rate of interest over the future time interval $[s, t]$ where $0 < s < t$. In this white paper we will construct an approximation of the forward rate curve using US Treasury Strip price quotes. To that end we will use the following hypothetical market data...

Our Hypothetical Problem

Assume the following discount factor curve at time zero...

Table 1: Market Discount Factors

Year	DFactor
0	1.0000
1	0.9500
2	0.8900
3	0.8200

Question 1: What are the continuous-time forward rates for years 1, 2 and 3?

Question 2: Construct the parabola equation to approximate the discount factor curve.

Question 3: Graph the continuous-time forward rate curve.

Build The Forward Rate Curve

We will defined the variable P_0 to be the time zero price of a treasury strip that matures at time t, the variable P_t to be value of the treasury strip at some future time t, and the variable D_t to be the discount factor over the time interval $[0, t]$. The equation for the price of a treasury strip at time zero is... [1]

$$P_0 = P_t\, D_t \text{ ...such that... } P_t = \frac{P_0}{D_t} \tag{1}$$

Note that if the treasury strip matures at time t then the equation for the value of the treasury strip at time t from the prespective of time zero is...

$$P_t = \text{Face value } \text{...given that... } t = \text{maturity date} \tag{2}$$

Using Equation (1) above and the spot rates in Table **??** above the future value of \$100 invested at time zero is worth the following amounts the end of years one, two and three...

$$P_1 = \frac{100.00}{0.9500} = 105.26 \text{ ...and... } P_2 = \frac{100.00}{0.8900} = 112.36 \text{ ...and... } P_3 = \frac{100.00}{0.8200} = 121.95 \tag{3}$$

We will define the variable $r_{s,t}$ to be the continuous-time forward rate over the time interval $[s, t]$. The equation for the price of a treasury strip at time t as a function of its price at time s is...

$$P_t = P_s \operatorname{Exp}\left\{ r_{s,t}\,(t - s) \right\} \tag{4}$$

Using Appendix Equation (16) below the equation for the continuous-time forward rate in Equation (3) above is...

$$r_{s,t} = \left[\ln\left(P_t\right) - \ln\left(P_s\right) \right] \Big/ \left(t - s\right) \tag{5}$$

Using Appendix Equation (17) below we can rewrite Equation (5) above as...

$$r_{s,t} = \Bigg[\ln\Bigg(D_s\Bigg) - \ln\Bigg(D_t\Bigg)\Bigg] \Bigg/ \Bigg(t - s\Bigg) \tag{6}$$

A more generalized version of Equation (6) above is...

$$r_{t,t+\Delta t} = \Bigg[\ln\Bigg(D_t\Bigg) - \ln\Bigg(D_{t+\Delta t}\Bigg)\Bigg] \Bigg/ \Delta t \tag{7}$$

We will define the variable f_t to be the continuous-time forward rate over the time interval $[t, t + \delta t]$. The forward rate is the limit of Equation (7) above as the change in time goes to zero, which is...

$$f_t = \lim_{\Delta t \to 0} r_{t,t+\Delta t} = -\frac{\delta \ln(D_t)}{\delta t} = -\frac{1}{D_t}\,\frac{\delta D_t}{\delta t} \tag{8}$$

Note that we modeled the discount factor in Equation (8) above via the following parabola equation... [1]

$$D_t = a\,t^2 + b\,t + c \;\; \text{...such that...} \;\; \frac{\delta D_t}{\delta t} = 2\,a\,t + b \tag{9}$$

Using Equation (9) above the solution to Equation (8) above is...

$$f_t = \lim_{\Delta t \to 0} r_{t,t+\Delta t} = -\frac{2\,a\,t + b}{a\,t^2 + b\,t + c} \tag{10}$$

The Answers To Our Hypothetical Problem

Question 1: What are the continuous-time forward rates for years 1, 2 and 3?

Using Equation (6) above the continuous-time forward rate for year one (i.e. time interval $[0, 1]$) is...

$$r_{0,1} = \frac{\ln(1.0000) - \ln(0.9500)}{1 - 0} = 0.0513 \tag{11}$$

Using Equation (6) above the continuous-time forward rate for year two (i.e. time interval $[1, 2]$) is...

$$r_{1,2} = \frac{\ln(0.9500) - \ln(0.8900)}{2 - 1} = 0.0652 \tag{12}$$

Using Equation (6) above the continuous-time forward rate for year three (i.e. time interval $[2, 3]$) is...

$$r_{2,3} = \frac{\ln(0.8900) - \ln(0.8200)}{3 - 2} = 0.0819 \tag{13}$$

Question 2: Construct the parabola equation to approximate the discount factor curve. [1]

Using market data in Table 1 above we will define the following matrix and vectors..

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 2.25 & 1.5 & 1 \\ 9 & 3 & 1 \end{bmatrix} \;\text{...and...}\; \vec{\mathbf{u}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \;\text{...and...}\; \vec{\mathbf{v}} = \begin{bmatrix} 1.0000 \\ 0.9200 \\ 0.8200 \end{bmatrix} \;\text{...such that...}\; \mathbf{A}\,\vec{\mathbf{u}} = \vec{\mathbf{v}} \tag{14}$$

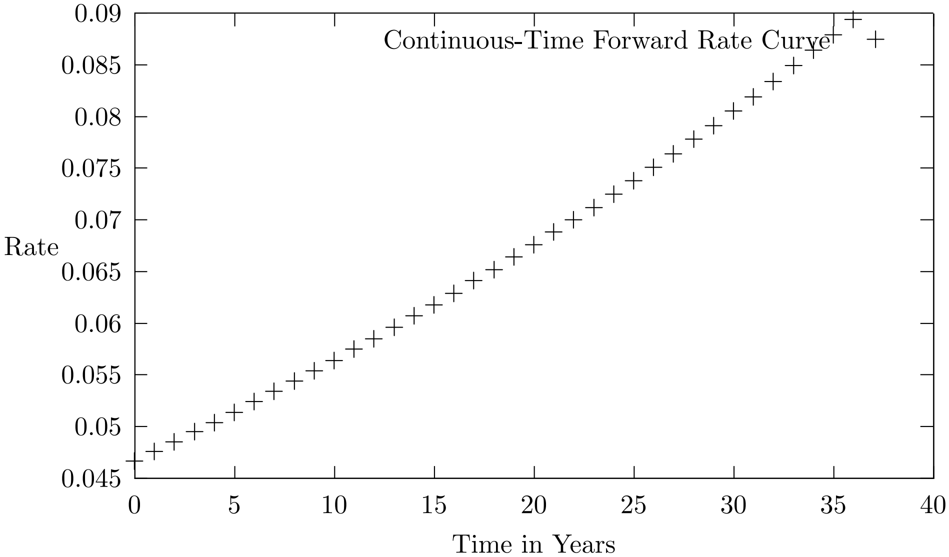
Note: The second row of the matrix $\mathbf{A}$ and vector $\vec{\mathbf{v}}$ above is an average of year one and year two prices.

Using the matrix and vector definitions in Equation (14) above we can solve for our parabola parameters as follows...

$$\mathbf{A}^{-1}\,\vec{\mathbf{v}} = \vec{\mathbf{u}} = \begin{bmatrix} -0.0044 \\ -0.0467 \\ 1.0000 \end{bmatrix} \tag{15}$$

Question 3: Graph the continuous-time forward rate curve.

Using Equation (10) above the graph of the continuous-time forward rate curve over the time interval $[0, 3]$ is...



References

[1] Gary Schurman, *The Spot Rate Curve,* February, 2019

Appendix

A. The following equation can be rewritten as...

$$P_t = P_s \,\mathrm{Exp}\left\{ r_{s,t}\left(t - s\right)\right\}$$

$$\ln\left(P_t\right) = \ln\left(P_s\right) r_{s,t}\left(t - s\right)$$

$$r_{s,t} = \left[\ln\left(P_t\right) - \ln\left(P_s\right)\right] \Big/ \left(t - s\right) \tag{16}$$

B. Using Equation (1) above the following equation can be rewritten as...

$$\ln\left(P_t\right) - \ln\left(P_s\right) = \ln\left(P_0\right) - \ln\left(D_t\right) - \ln\left(P_0\right) + \ln\left(D_s\right)$$

$$= \ln\left(D_s\right) - \ln\left(D_t\right) \tag{17}$$